GENERATING HIERARCHIAL SCALE FREE GRAPHS FROM FRACTALS

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Abstract. Motivated by the hierarchial network model of E. Ravasz, A.-L. Barabási, and T. Vicsek [3] and [2], we introduce deterministic scale-free networks derived from a graph directed self-similar fractal Λ. With rigorous mathematical results we verify that our model captures some of the most important features of many real networks: the scale free and the high clustering properties. We also prove that the diameter is the logarithm of the size of the system. Using our (deterministic) fractal Λ we generate random graph sequence sharing similar properties.

1. Introduction

In the last two decades there have been a considerable amount of attention paid to the study of complex networks like the World Wide Web, social networks, or biological networks. This resulted in the construction of numerous network models, see e.g. [1], [9], [7], [4], [10] [5]. Most of them use a version of preferential attachment and are of probabilistic nature. A completely different approach was initiated by Barabási, Ravasz, and Vicsek [3]. They introduced deterministic network models generated by a method which is common in constructing fractals. Their model exhibits hierarchical structure and the degree sequence obeys power law decay. To model also the clustering behavior of real networks, Ravasz and Barabási [2] developed the original model so that their deterministic network model preserved the same power law decay and has similar clustering behavior to many real networks. Namely, the average local clustering coefficient is independent of the size of the network and the local clustering coefficient decays inversely proportional to the degree of the node.

In this paper we generalize both of the models above. Starting from an arbitrary initial bipartite graph G on N vertices, we construct a hierarchical sequence of deterministic graphs G_n. Namely, $V(G_n)$, the

2000 *Mathematics Subject Classification.* Primary 05C80 Secondary 28A80

Key words and phrases. Random graphs, Scale-free networks;

The research was supported by the NKTH OTKA grant # 7778.

set of vertices of G_n is $\{0, 1, \dots, N-1\}^n$. To construct G_n from G_{n-1}, we take N identical copies of G_{n-1}, each of them identified with a vertex of G. Then we connect these components in a complicated way described in (1). In this way, G_n contains N^{n-1} copies of G_1, which are connected in a hierarchical manner, see Figures 1(a), 1(b) and 3 for two examples.

There are no triangles in G_n. Hence, in order to model the clustering properties of many real networks, we need to extend the set of edges of our graph sequence to destroy the bipartite property. Motivated by [2], we add some additional edges to G_1 to obtain the (no longer bipartite) graph $\widehat{G}_1$. Then we build up the graph sequence $\widehat{G}_n$ as follows: $\widehat{G}_n$ consist of N^{n-1} copies of $\widehat{G}_1$, which copies are connected to each other in the same way as they were in G_n. So, $\widehat{G}_n$ and G_n have the same vertex set and their edges only differ at the lowest hierarchical level, that is, within the N^{n-1} copies of G_1 and $\widehat{G}_1$, see Figures 3 and 4. We give a rigorous proof of the fact that the average local clustering coefficient of $\widehat{G}_n$ does not depend on the size and the local clustering coefficient of a node with degree k is of order $1/k$.

The embedding of the adjacency matrix of the graph sequence G_n is carried out as follows: A vertex $\underline{x} = (x_1 \dots x_n)$ is identified with the corresponding N-adic interval $I_{\underline{x}}$ (see (4)). Λ_n is the union of those $N^{-n} \times N^{-n}$ squares $I_{\underline{x}} \times I_{\underline{y}}$ for which the vertices $\underline{x}, \underline{y}$ are connected by an edge in G_n. So, Λ_n is the most straightforward embedding of the adjacency matrix of G_n into the unit square. Λ_n turns out to be a nested sequence of compact sets, which can be considered as the n-th approximation of a graph directed self-similar fractal Λ on the plane, see Figure 1(c). We discuss connection between the graph theoretical properties of G_n and properties of the limiting fractal Λ.

Furthermore, using Λ we generate a random graph sequence G_n^{r} in a way which was inspired by the W-random graphs introduced by Lovász and Szegedy [10]. See also Diaconis, Janson [6], which paper contains a list of corresponding references. We show that the degree sequence has power law decay with the same exponent as the deterministic graph sequence G_n. Thus we can define a random graph sequence with a prescribed power law decay in a given range. Bollob'as, Janson and Riordan [5] considered inhomogeneous random graphs generated by a kernel. Our model is not covered by their construction, since Λ is a fractal set of zero two dimensional Lebesgue measure.

The paper is organized as follows: In Section 2 we define the deterministic model and the associated fractal set Λ. In Section 3, we verify the

scale free property of G_n (Theorem 3.1). We compare the Hausdorff dimension of Λ to the power law exponent of the degree sequence of G_n. Our next result is that both of the diameter of G_n and the average length of shortest path between two vertices are of order of the logarithm of the size of G_n (Corollary 3.6 and Theorem 3.7). In Section 3.4 we prove the above mentioned properties of the clustering coefficient of $\widehat{G}_n$ (Theorem 3.13 and 3.11). In Section 4 we describe the randomized model, and in Section 5 we prove that the model exhibits the same power law decay as the corresponding deterministic version.

2. Deterministic model

The model was motivated by the hierarchical graph sequence model in [3], and is given as follows.

2.1. **Description of the model.** Let G, our *base graph*, be any labeled bipartite graph on the vertex set $\Sigma_1 = \{0, \dots, N-1\}$. We partition Σ_1 into the non-empty sets V_1, V_2 and one of the end points of any edge is in V_1, and the other is in V_2. We write $n_i := |V_i|$, $i = 1, 2$ for the cardinality of V_i. The edge set of G is denoted by $E(G)$. If the pair $x, y \in \Sigma_1$ is connected by an edge, then this edge is denoted by $\binom{x}{y}$, since this notation makes it convenient to follow the labels of the vertices along a path.
Now we define our graph sequence $\{G_n\}_{n\in\mathbb{N}}$ generated by the base graph G.
The vertex set is $\Sigma_n = \{(x_1 x_2 \dots x_n) : x_i \in \Sigma_1\}$, all words of length n above the alphabet Σ_1. To be able to define the edge set, we need some further definitions.

Definition 2.1.

(1) *We assign a type to each element of* Σ_1. *Namely,*

$$typ(x) = \begin{cases} 1, & \text{if } x \in V_1; \\ 2, & \text{if } x \in V_2. \end{cases}$$

(2) *We define the* **type** *of a word* $\underline{z} = (z_1 z_2 \dots z_n) \in \Sigma_n$ *as follows: if all the elements* $z_j, j = 1, \dots, n$ *of* $\underline{z}$ *fall in the same* V_i, $i = 1, 2$ *then* $typ(\underline{z})$ *the type of* $\underline{z}$ *is* i. *Otherwise* $typ(\underline{z}) := 0$.

(3) *For* $\underline{x} = (x_1 \dots x_n), \underline{y} = (y_1 \dots y_n) \in \Sigma_n$ *we denote the* **common prefix** *by*

$$\underline{x} \wedge \underline{y} = (z_1 \dots z_k) \text{ s.t. } x_i = y_i = z_i, \forall i = 0, \dots, k \text{ and } x_{k+1} \neq y_{k+1}.$$

(4) *Given* $\underline{x} = (x_1 \dots x_n), \underline{y} = (y_1 \dots y_n) \in \Sigma_n$, *the* **postfixes** $\tilde{\underline{x}}, \tilde{\underline{y}} \in \Sigma_{n-|\underline{x}\wedge\underline{y}|}$ *are determined by*

$$\underline{x} = (\underline{x} \wedge \underline{y})\tilde{\underline{x}}, \ \underline{y} = (\underline{x} \wedge \underline{y})\tilde{\underline{y}},$$

where the concatenation of the words $\underline{a}, \underline{b}$ *is denoted by* $\underline{ab}$.

Now we can define the edge set $E(G_n)$. Two vertices $\underline{x}$ and $\underline{y}$ in G_n are connected by an edge if and only if the following assumptions hold:

(a): One of the postfixes $\tilde{\underline{x}}, \tilde{\underline{y}}$ is of type 1, the other is of type 2,

(b): for each $i > |x \wedge y|$, the coordinate pair $\binom{x_i}{y_i}$ forms an edge in G.

That is, $E(G_n) \subset \Sigma_n \times \Sigma_n$:

$$E(G_n) = \left\{ \binom{\underline{x}}{\underline{y}} \Big| \underline{x} = \underline{y} \text{ or } \right.$$

$$\left. \{\mathrm{typ}(\tilde{\underline{x}}), \mathrm{typ}(\tilde{\underline{y}})\} = \{1,2\}, \forall |\underline{x} \wedge \underline{y}| < i \le n, \binom{x_i}{y_i} \in E(G) \right\} \tag{1}$$

Remark 2.2. *Note that we artificially added all loops to the (otherwise bipartite) graph sequence* G_n*, implying easier calculations later without loss of the important properties. In particular,* G_1 *differs from* G *only in the loops.*

Remark 2.3 (Hierarchical structure of G_n)**.** *For every initial digit* $x \in \{0, 1, \dots, N-1\}$*, consider the set* W_x *of vertices* $(x_1 \dots x_n)$ *of* G_n *with* $x_1 = x$*. Then the induced subgraph on* W_x *is identical to* G_{n-1}*.*

We write $\deg_n(\underline{x})$ for the *degree of a vertex* in G_n, including the loop which increases the degree by 2. However, for an $x \in \Sigma_1$, $\deg x$ denotes degree of x in G. In particular $\deg_1(x) = \deg(x)+2$. In what follows, we will frequently use $\ell(\underline{x})$, the length of the longest block from backwards in $\underline{x}$ which has a nonzero type,

$$\ell(\underline{x}) := \max_{i \in \mathbb{N}} \{\mathrm{typ}(x_{n-i+1}, \dots x_n) \in \{1,2\}\} \tag{2}$$

Remark 2.4. *The degree of a node* $\underline{x} \in \Sigma_n$

$$\deg_n(\underline{x}) = 2 + S(\underline{x}) \cdot \deg(x_n),$$

where

$$S(\underline{x}) : = 1 + \deg(x_{n-1}) + \dots + \deg(x_{n-1}) \deg(x_{n-2}) \cdots \deg(x_{n-\ell(\underline{x})+1})$$

$$= \sum_{r=0}^{\ell(\underline{x})-1} \left(\prod_{j=1}^{r} \deg(x_{n-j}) \right), \tag{3}$$

where the empty sum is meant to be 1.

The following two examples satisfy the requirements of our general model.

Example 2.5 (Cherry)**.** *Barabási, Ravasz and Vicsek* [3] *introduced the "cherry" model presented on Figures 1(a) and 1(b): Let* $V_1 = \{1\}$ *and* $V_2 = \{0, 2\}$*,* $E(G) = \{(1, 0), (1, 2)\}$*.*

Example 2.6 (Fan)**.** *Our second example is called "fan", and is defined on Figure 3. Note that here* $|V_1| > 1$*.*

2.2. **The embedding of the adjacency matrices into** $[0, 1]^2$**.** In this Section, we investigate the sequence of adjacency matrices corresponding to $\{G_n\}_{n\in\mathbb{N}}$. Roughly speaking, we will map them in the unit square, see Figure 1(c).
To represent the adjacency matrix of G_n as a subset of the unit square, first partition $[0, 1]^2$ into N^{2n} congruent boxes, i.e. divide $[0, 1]$ into equal subintervals of length $\frac{1}{N^n}$, corresponding to the first n digits of the N-adic expansion of elements of $[0, 1]$:

$$I_{x_1\dots x_n} = \left[\sum_{r=1}^{n} \frac{x_r}{N^r}, \sum_{r=1}^{n} \frac{x_r}{N^r} + \frac{1}{N^n}\right], \forall (x_1 \dots x_n) \in \Sigma_n. \tag{4}$$

We partition $[0, 1]^2$ with the corresponding level-n squares:

$$Q_{\binom{\underline{x}}{\underline{y}}} := I_{\underline{x}} \times I_{\underline{y}}, \quad \binom{\underline{x}}{\underline{y}} \in \Sigma_n \times \Sigma_n. \tag{5}$$

A natural embedding of the adjacency matrix of G_n in the unit square is as follows:

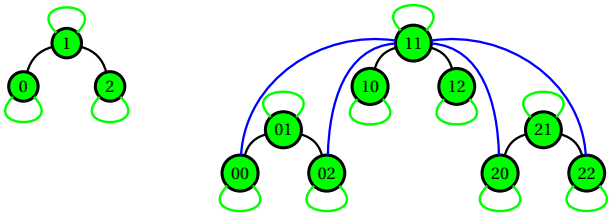
$$\Lambda_n(a, b) := \begin{cases} 1, & \text{if } (a, b) \in Q_{\binom{\underline{x}}{\underline{y}}}, \binom{\underline{x}}{\underline{y}} \in E(G_n); \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

That is,

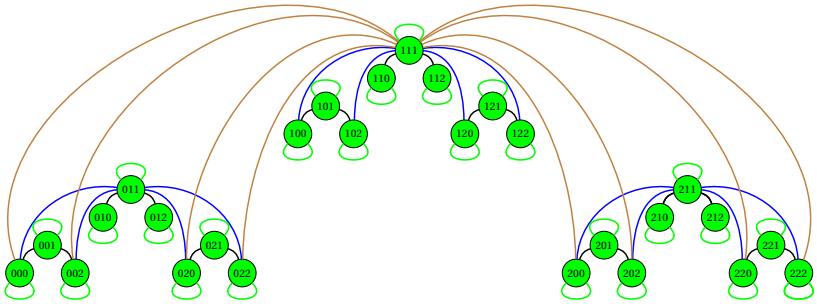
$$\Lambda_n(a, b) = \sum_{\substack{\underline{x},\underline{y}\in\Sigma_n \\ \binom{\underline{x}}{\underline{y}}\in E(G_n)}} \mathbb{1}_{Q_{\binom{\underline{x}}{\underline{y}}}}(a, b).$$

We write Λ_n for the support of the function $\Lambda_n(a, b)$, see Figure 1(c). Observe that Λ_n is a compact set and $\Lambda_{n+1} \subset \Lambda_n$ holds for all n. So we can define the non-empty compact set

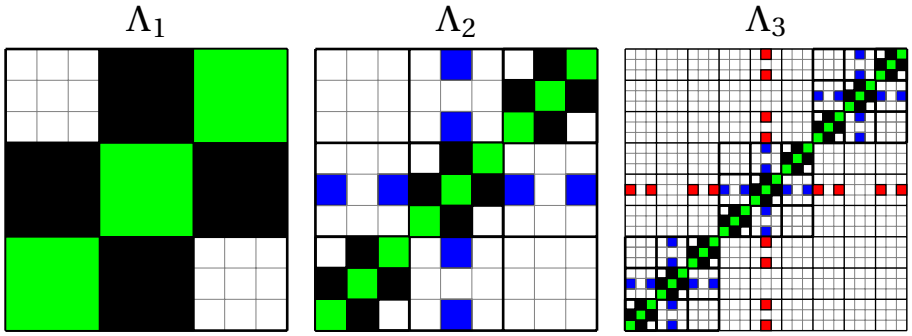
$$\Lambda := \bigcap_{n=1}^{\infty} \Lambda_n. \tag{7}$$



(a) G_1 and G_2 with loops



(b) G_3



(c) The sets $\Lambda_1, \Lambda_2, \Lambda_3$

FIGURE 1. $G_1, G_2, G_3, \Lambda_1, \Lambda_2, \Lambda_3$ for the cherry Example 2.5

Clearly,

$$\mathbb{1}_\Lambda(a,b) = \lim_{n\to\infty} \Lambda_n(a,b).$$

Remark 2.7. *This representation obviously depends on the labeling of the graph G. For an arbitrary permutation π of $\{0,\dots,N-1\}$, the corresponding representation of G_n is denoted by $\Lambda_n^\pi(a,b)$. The relation*

between these two representations is given by the formula

$$\Lambda_n^\pi(a,b) = \Lambda_n(\varphi_{\pi^{-1}}(a), \varphi_{\pi^{-1}}(b)), \text{ and}$$
$$\mathbb{1}_{\Lambda^\pi}(a,b) = \mathbb{1}_\Lambda(\varphi_{\pi^{-1}}(a), \varphi_{\pi^{-1}}(b)),$$

where the measurable function $\varphi_\pi(x) : [0,1] \to [0,1]$ *is defined by*

$$\varphi_\pi\left(\sum_{i=1}^{\infty} \frac{x_i}{N^i}\right) = \sum_{i=1}^{\infty} \frac{\pi(x_i)}{N^i}.$$

2.3. **Graph-directed structure of Λ.** Now we prove that the limit Λ (defined in (7)) can be considered as the attractor of a not irreducible graph-directed self-similar iterated function system, (for the definition see [8]), with the directed graph $\mathcal{G}$ defined below.

Definition 2.8. *The* **vertex set** $V(\mathcal{G})$ *is partitioned into three subsets:*

$$V_{dd} = \left\{ \binom{z}{z}, z \in \Sigma_1 \right\}$$
$$V_{12} = \left\{ \binom{x}{y}, x \in V_1, y \in V_2 \right\} \tag{8}$$
$$V_{21} = \left\{ \binom{x}{y}, x \in V_2, y \in V_1 \right\}.$$

Then

$$V(\mathcal{G}) = V_{dd} \cup V_{12} \cup V_{21}.$$

The **set of directed edges** $E(\mathcal{G})$ *of* $\mathcal{G}$ *is as follows: First we connect all vertices in both directions within each of the three sets* V_{dd}, V_{12} *and* V_{21} *(loops included). Then there is an outgoing edge for each vertex in* V_{dd} *to all vertices in* V_{12} *and* V_{21}.
For every directed edge $e = (v_1, v_2) \in E(\mathcal{G})$ *we define a homothety:*

$$f_e : Q_{v_2} \to Q_{v_1}, \ f_e(a,b) := \frac{1}{N}(a,b) + \frac{1}{N}(x_1, y_1), \text{ with } v_i = \binom{x_i}{y_i}, \tag{9}$$

where $Q_v := Q_{\binom{x}{y}}$ *is the level-1 square for* $v = \binom{x}{y} \in V(\mathcal{G})$.

The graph $\mathcal{G}$ corresponding to the graph sequence in the "cherry" example is given by Figure 2.
In general, $\mathcal{G}$ is given by the schematic picture on the right hand side of Figure 2, where the double arrow in between the complete directed graphs $\overleftrightarrow{K_\cdot(V_{\cdot\cdot})}$ illustrates that we connect all pairs of vertices in the given direction.
Let $\mathcal{P}_n$ be the set of all paths of length n in $\mathcal{G}$, i.e.

$$\mathcal{P}_n := \{\underline{v} = (v_1 \dots v_n) | \forall\ 1 \le i < n\ (v_i, v_{i+1}) \in E(\mathcal{G})\}.$$

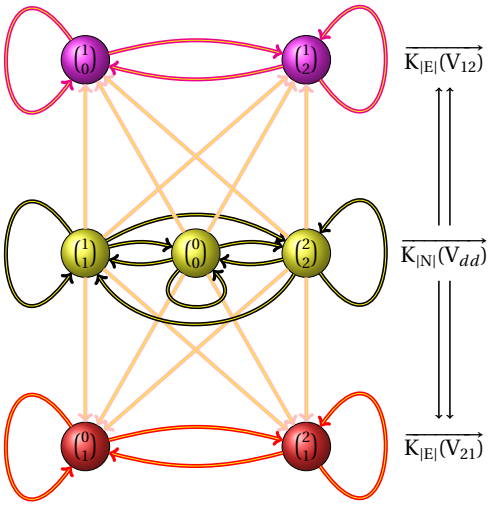


FIGURE 2. The graph $\mathcal{G}$ for the "cherry", Example 2.5.

For a $\underline{v} = (v_1 \dots v_n) = \binom{x_1 \dots x_n}{y_1 \dots y_n} \in \mathcal{P}_n$ it immediately follows from definitions (5) and (9) that

$$Q_{\underline{v}} = f_{\underline{v}}\left([0,1]^2\right) = I_{x_1 \dots x_n} \times I_{y_1 \dots y_n}, \tag{10}$$

where

$$\begin{aligned} f_{\underline{v}}(.) &:= f_{(v_1,v_2)} \circ \dots \circ f_{(v_{n-1},v_n)}(.) \text{ if } n \geq 2, \\ f_v(a,b) &:= \frac{1}{N}(a,b) + \frac{1}{N}(x,y), \text{ if } n = 1,\ v = \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned} \tag{11}$$

The key observation of connecting $\mathcal{G}$ to the graph sequence G_n is the following:

Claim 2.9. *For all n we have*

$$E(G_n) = \mathcal{P}_n.$$

Proof. Let $\underline{v} = (v_1 \dots v_n) = \binom{a_1 \dots a_n}{b_1 \dots b_n} \in \Sigma_n \times \Sigma_n$, thus $\underline{a} = (a_1 \dots a_n)$ and $\underline{b} = (b_1 \dots b_n)$ are vertices in G_n. First we assume that $\underline{v} \in E(G_n)$. Observe that by (1), $\binom{a_i}{b_i}$ are vertices in $\mathcal{G}$. We would like to prove that the sequence

$$\binom{a_1}{b_1} \dots \binom{a_n}{b_n} \in \mathcal{P}_n. \tag{12}$$

If $k := |\underline{a} \wedge \underline{b}| \geq 1$, then for $i \leq k$, $a_i = b_i$ holds, thus the sequence of points $\binom{a_1}{b_1} \dots \binom{a_k}{b_k}$ forms a path in $\overrightarrow{K_{|N|}(V_{dd})}$. By (1), the pairs $\binom{a_{k+1}}{b_{k+1}}, \dots, \binom{a_n}{b_n}$ are all edges in G thus vertices in $\mathcal{G}$. Furthermore, either they all belong to V_{12} or they are all contained in V_{21}, see (8). This implies that this postfix also forms a path in $\overrightarrow{K_{|N|}(V_{12})}$ or in $\overrightarrow{K_{|N|}(V_{21})}$. By definition of $E(\mathcal{G})$, $\left(\binom{a_k}{b_k}, \binom{a_{k+1}}{b_{k+1}}\right)$ is an edge in $\mathcal{G}$, so $\binom{a_1}{b_1} \dots \binom{a_n}{b_n}$ is a path in $\mathcal{G}$. If $k = 0$ then the whole path is contained either in V_{12} or in V_{21}. This completes the proof of (12).
On the other hand, if $\binom{a_1}{b_1} \dots \binom{a_n}{b_n}$ is a path of length n in $\mathcal{G}$, then we claim that for $\underline{a} = (a_1 \dots a_n), \underline{b} = (b_1 \dots b_n) \in V(G_n)$

$$(\underline{a}, \underline{b}) \in E(G_n).$$

The proof is very similar to the previous one. □

In this way we can characterize Λ_n as follows:

Corollary 2.10.

$$\Lambda_n = \bigcup_{\underline{v} \in \mathcal{P}_n} Q_{\underline{v}} = \bigcup_{\underline{v} \in \mathcal{P}_n} f_{\underline{v}}\left([0,1]^2\right).$$

Proof. Immediately follows from (6) and (10) and the assertion of the Claim 2.9. □

Let us define

$$\mathcal{P}_\infty := \{\underline{v} = (v_1 v_2 \dots) | \forall i \in \mathbb{N},\ (v_i, v_{i+1}) \in E(\mathcal{G})\}.$$

Now for every $\underline{v} \in \mathcal{P}_\infty$ we have $\bigcap_{n=1}^{\infty} Q_{(v_1 \dots v_n)}$ is a point in $[0,1]^2$, which will be denoted by $\Pi_{\underline{v}}$. That is,

$$\Pi : \mathcal{P}_\infty \to [0,1]^2,\ \Pi(\underline{v}) := \bigcap_{n=1}^{\infty} Q_{(v_1 \dots v_n)} = \lim_{n \to \infty} f_{v_1 \dots v_n}(0,0).$$

It is an immediate consequence of Corollary 2.10, that

$$\Pi(\mathcal{P}_\infty) = \Lambda,\ \text{i.e. } \Lambda = \bigcup_{\underline{v} \in \mathcal{P}_\infty} \Pi_{\underline{v}}. \tag{13}$$

This means that Λ_n, the embedded adjacency matrix of G_n, can be considered as the n-th approximation of the fractal set Λ.
In this way we coded the elements of Λ by the elements of $\mathcal{P}_\infty$. This coding is not $1-1$ for the same reason as the N-adic expansion is not $1-1$. However, if neither of the two coordinates of a point $(a,b) \in \Lambda$ are N-adic rational numbers, then (a,b) has a unique code.

2.4. **Fractal geometric characterisation of Λ.** For notational convenience we define the set of finite words above the alphabet V_{dd} (including the empty word as well):

$$V_{dd}^* := \{\underline{v}|\ \exists n \in \mathbb{N} \cup \{0\}\,, \underline{v} = (v_1 \dots v_n) \text{ and } v_i \in V_{dd}\}\,.$$

The three subgraphs $\overrightarrow{K_{|E|}(V_{12})}$, $\overrightarrow{K_{|E|}(V_{21})}$ and $\overrightarrow{K_{|E|}(V_{dd})}$ of $\mathcal{G}$ are complete directed graphs. We consider the three corresponding self-similar iterated function systems (IFS):

$$\begin{aligned}\mathcal{F}_{dd} &:= \{f_v\}_{v\in V_{dd}}\,,\\ \mathcal{F}_{12} &:= \{f_v\}_{v\in V_{12}}\,,\\ \mathcal{F}_{21} &:= \{f_v\}_{v\in V_{21}}\,,\end{aligned}$$

where the functions $f_v, v \in V(\mathcal{G})$ were defined in (11). The attractors of these IFS-s (see [8, p.30]) are the unique nonempty compact sets satisfying

$$\begin{aligned}\Lambda_{dd} &:= \bigcup_{v\in V_{dd}} f_v(\Lambda_{dd}) = \{\Pi(\underline{v})|\underline{v} = (v_1, v_2 \dots) \text{ and } v_i \in V_{dd}\}\\ \Lambda_{12} &:= \bigcup_{v\in V_{12}} f_v(\Lambda_{12}) = \{\Pi(\underline{v})|\underline{v} = (v_1, v_2 \dots) \text{ and } v_i \in V_{12}\}\\ \Lambda_{21} &:= \bigcup_{v\in V_{21}} f_v(\Lambda_{21}) = \{\Pi(\underline{v})|\underline{v} = (v_1, v_2 \dots) \text{ and } v_i \in V_{21}\}\,.\end{aligned} \tag{14}$$

The Open Set Condition (see e.g. [8, p.35]) holds for these IFS-s, so we can easily compute the Hausdorff-dimension of the attractors. Clearly, Λ_{dd} is the diagonal of the unit square.
Now we prove that Λ is a countable union of homothetic copies of these attractors.

Theorem 2.11.

$$\Lambda = \underbrace{\text{Diag}}_{\Lambda_{dd}} \cup \bigcup_{\underline{v}\in V_{dd}^*} (f_{\underline{v}}(\Lambda_{12}) \cup f_{\underline{v}}(\Lambda_{21}))\,,$$

where $\text{Diag} = \{(x, x) : x \in [0, 1]\}$.

Remark 2.12. *Observe that Λ_{21} is the image of Λ_{12} by the reflection through the diagonal, hence Λ is symmetric to the diagonal. The same is true for the n-th approximation Λ_n of Λ. This can be seen immediately by using the embedded adjacency matrix characterization of Λ_n.*

Proof of Theorem 2.11. We start by showing that

$$\Lambda \subset \text{Diag} \cup \bigcup_{\underline{v}\in V_{dd}^*} (f_{\underline{v}}(\Lambda_{12}) \cup f_{\underline{v}}(\Lambda_{21}))\,. \tag{15}$$

Pick an arbitrary point $(a,b) \in \Lambda$. As a consequence of (13) there exists a $\underline{v} = (v_1 v_2 \dots) \in \mathcal{P}_\infty$ such that $\Pi(\underline{v}) = (a,b)$. Let $k := \max\{\ell : v_\ell \in \Lambda_{dd}\}$. We distinguish three cases: $k = 0$, $k = \infty$ or $0 < k < \infty$. Mind that for all $i \le k, v_i \in V_{dd}$ since once the path left the component V_{dd}, there is no way to return. Since V_{12} and V_{21} are closed, for $k < \infty$ all $v_i, i > k$ are in the same component V_{12} or V_{21}.

Case $k = 0$**:** Clearly either all v_i are in V_{12} or in V_{21}, so $\Pi(\underline{v}) \in \Lambda_{12} \cup \Lambda_{21}$.

Case $k = \infty$**:** For the same reason, $\Pi(\underline{v}) = \lim_{n\to\infty} f_{v_1 \dots v_n}(0,0) \in \Lambda_{dd} = \mathrm{Diag}$. This is so because $f_{v_1 \dots v_n}(0,0)$ is in the $\frac{1}{N^n}$ neighborhood of the diagonal $\{(x,x) : x \in [0,1]\}$.

Case $0 < k < \infty$**:** Let $\underline{v}_k = (v_1 \dots v_k)$. For symmetry, without loss of generality we may assume that $v_{k+1} \in V_{12}$. As in the first case, we can see that for $\underline{w} := (v_{k+1} v_{k+2} \dots)$, $\Pi(\underline{w}) \in \Lambda_{12}$. Hence $\Pi(\underline{v}) = f_{\underline{v}_k}(\Pi_{\underline{w}}) \in f_{\underline{v}_k}(\Lambda_{12})$.

Now we have verified (15). To prove the opposite direction, that is

$$\Lambda \supset \mathrm{Diag} \cup \bigcup_{\underline{v} \in V_{dd}^*} \left(f_{\underline{v}}(\Lambda_{12}) \cup f_{\underline{v}}(\Lambda_{21}) \right), \tag{16}$$

we will use the symbolic representation of Λ given in (13).
Pick an $x \in [0,1]$ and take the N-adic code $(x_1 x_2 \dots)$ of x. That is, $x = \sum_{n=1}^{\infty} \frac{x_i}{N^i}$, $x_i \in \{0, \dots, N-1\}$. Then

$$\underline{v} := \left(\underbrace{\begin{pmatrix} x_1 \\ x_1 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} x_2 \\ x_2 \end{pmatrix}}_{v_2}, \dots \right) \in \mathcal{P}_\infty,$$

it is easy to see that $\Pi(\underline{v}) = (x,x)$. So by (13), $(x,x) \in \Lambda$.
Now we assume that $(a,b) \in \bigcup_{\underline{v} \in V_{dd}^*} (f_{\underline{v}}(\Lambda_{12}) \cup f_{\underline{v}}(\Lambda_{21}))$. Without loss of generality we may further assume that $(a,b) \in f_{\underline{v}}(\Lambda_{12})$ for some $\underline{v} \in V_{dd}^*$. That is, $(a,b) = f_{\underline{v}}(a',b')$ where $(a',b') \in \Lambda_{12}$. By (14) there exists a $\underline{w} := (w_1 w_2 \dots)$, $w_i \in V_{12}$ such that $\Pi(\underline{w}) = (a',b')$. In this way, for the concatenation $\underline{t} := \underline{v}\underline{w} \in \mathcal{P}_\infty$ we have $(a,b) = \Pi(\underline{t})$ which implies $(a,b) \in \Lambda$. This completes the proof of (16). □

2.5. **The same model without loops.** Let G'_n be the same graph as G_n but without loops, i.e. $V(G'_n) = V(G_n)$ and $E(G'_n) \subset \Sigma_n \times \Sigma_n$:

$$E(G'_n) = \left\{ \binom{\underline{x}}{\underline{y}} \quad \middle| \quad \{\mathrm{typ}(\tilde{\underline{x}}), \mathrm{typ}(\tilde{\underline{y}})\} = \{1,2\} \text{ and } \forall |\underline{x} \wedge \underline{y}| < i \leq n, \binom{x_i}{y_i} \in E(G) \right\}$$

In this case $\Lambda'_n = \Lambda_n \setminus \mathrm{Diag}_n$, where Diag_n is the union of the level n squares that have nonempty intersection with the diagonal. The sequence Λ'_n is not a nested sequence of compact sets. However, it is easy to see that the characteristic function of Λ'_n tends to characteristic function of $\Lambda \setminus \mathrm{Diag}$. Further, Λ'_n tends to Λ in the Hausdorff metric, see [8].

3. Properties of the sequence $\{G_n\}$ and Λ

In this section we compute the degree distribution of G_n, and relate it to the Hausdorff dimension of Λ. We also compute the length of the average shortest path in G_n. To get interesting result about the local clustering coefficient we need to modify our graph sequence G_n in the line as it was done in [2].

3.1. **Degree distribution of $\{G_n\}$.** Here we compute the degree distribution under the following regularity assumption on the base graph G:

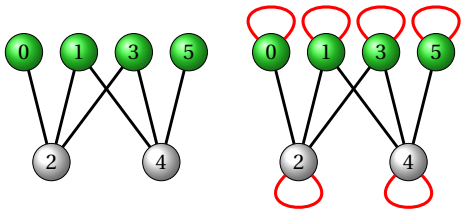
$$\textbf{(A1)} \qquad \begin{aligned} &\deg(x) := d_1, \quad \forall x \in V_1 \\ &\max_{j \in V_2} \deg(y) := d_2 \leq d_1 - 1, \quad \forall y \in V_2 \end{aligned}$$

Recall that we defined $\ell(\underline{x})$ in (2) as the length of the longest block from backwards of the node $\underline{x}$ such that the last $\ell(\underline{x})$ digits of $\underline{x}$ belong to the same V_i. Put $\Sigma_n^i := \{\underline{x} \in \Sigma_n | x_n \in V_i\}, i = 1, 2$. It follows from **A1** and Remark 2.4 that the degree of a node $\underline{x} \in \Sigma_n^1$ is $\frac{d_1^{\ell(\underline{x})+1}-1}{d_1-1} + 1$, and the number of such nodes with $\ell(\underline{x}) = \ell$ is exactly $N^{n-\ell+1} \cdot n_2 \cdot n_1^\ell$. Under assumption **A1**, the decay of the degree distribution is determined by the set of high degree nodes denoted by

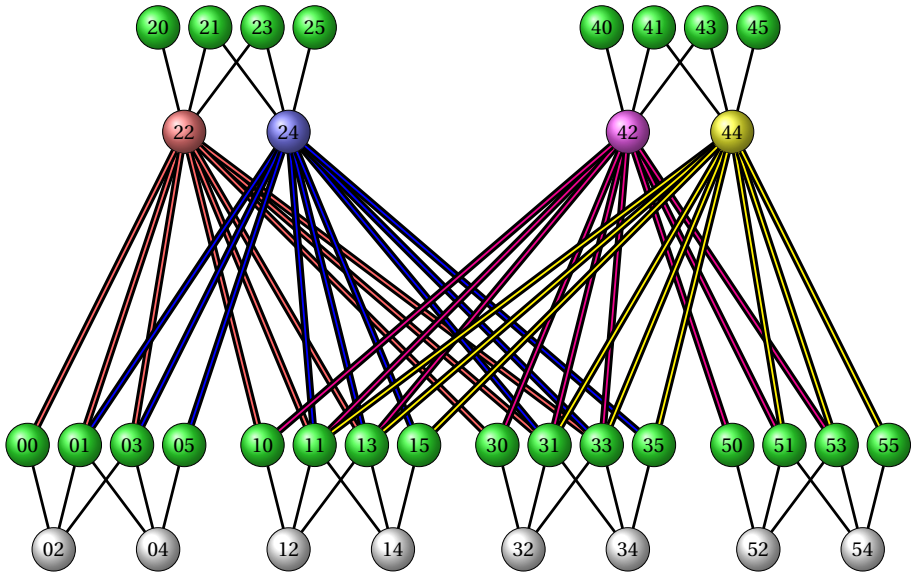
$$HD_n := \left\{ \underline{x} \in \Sigma_n^1 | \deg_n(\underline{x}) > \max_{\underline{y} \in \Sigma_n^2} \deg_n(\underline{y}) \right\}.$$

An equivalent characterisation of HD_n is

$$HD_n = \left\{ \underline{x} \in \Sigma_n^1 | \ell(\underline{x}) > \frac{1}{\log d_1} \max\left\{ (n+1)\log(d_2), \log n \right\} \right\}.$$



(a) G on the left and G_1 on the right hand side. Here $V_1 = \{2, 4\}$ and $V_2 = \{0, 1, 3, 5\}$



(b) The graph G_2 (contains additionally all loops).

FIGURE 3. Example "fan".

This is so because the degree of any $\underline{y} \in \Sigma_n^2$ is at most $\max\{d_2^{n+1}, n\}$. The tail of the cumulative degree distribution is

$$\mathbb{P}\left[\deg_n(\underline{X}) > \frac{d_1^{\ell+1} - 1}{d_1 - 1} + 1\right] = \frac{n_1^{\ell+1} N^{n-\ell-1}}{N^n} = \left(\frac{n_1}{N}\right)^{\ell+1},$$

where $\underline{X}$ is a uniformly chosen node of G_n. Mind that as long as $\ell < n$, this probability does not depend on n. Writing $\widetilde{F}(t) = \mathbb{P}(\deg(\underline{X}) > t)$ for the tail of the cumulative distribution function we get the power law decay

$$\widetilde{F}(t) = t^{-\frac{\log(N/n_1)}{\log d_1}} \cdot c(d_1) \quad \text{for } t = \frac{d_1^{\ell+1} - 1}{d_1 - 1}.$$

So we have proved

Theorem 3.1. *The degree distribution of the graph sequence G_n satisfying assumption **A1**, has a power law decay with exponent*

$$\tilde{\gamma} = \gamma - 1 = \frac{\log(N/n_1)}{\log d_1}. \tag{17}$$

This implies that the largest decay γ we can get in this family of models is $1 + \frac{\log 3}{\log 2}$, and the maximum is attained at $n_1 = 1$ and $d_1 = 2 = n_2$. This is exactly the graph sequence in Example 2.5, see Figures 1(a) and 1(b). We will later see that the case $n_1 = 1$ is important in another sense as well, see Section 3.2.

3.2. **Hausdorff dimension of Λ.** In Theorem 2.11 we decomposed Λ into the diagonal of the square and countably many homothetic copies of Λ_{12} and Λ_{21}, both attractors of self-similar IFS-s. Hence the Hausdorff dimension is the maximum of the dimension of the diagonal and $\dim_{\mathrm{H}} \Lambda_{12} = \dim_{\mathrm{H}} \Lambda_{21}$. Note that the self-similar IFS $\mathcal{F}_{12}$ consists of $|E|$ similarities of contraction ratio $\frac{1}{N}$, and satisfies the Open Set Condition. As an immediate application of [8, Theorem 2.7], the Hausdorff dimension of Λ_{12} and Λ_{21} is $\dim_{\mathrm{H}} \Lambda_{12} = \frac{\log |E|}{\log N}$.
By this argument above we have proved the following theorem:

Theorem 3.2. *The Hausdorff dimension of Λ is*

$$\dim_{\mathrm{H}} \Lambda = \max\left\{ \frac{\log |E|}{\log N}, 1 \right\},$$

furthermore,

$$\dim_{\mathrm{H}} \big(\Lambda \backslash \mathrm{Diag}\big) = \frac{\log |E|}{\log N}.$$

Corollary 3.3. *If $|V_1| = n_1 = 1$, then (**A1**) holds with $d_1 = |E|$ in the bipartite G. Hence the degree distribution exponent (17) equals*

$$\tilde{\gamma} = \frac{\log N}{\log |E|} = \frac{1}{\dim_{\mathrm{H}} \big(\Lambda \backslash \mathrm{Diag}\big)}.$$

3.3. **Average shortest path in G_n.** In many real networks, the typical distance between two randomly chosen points is of order $\log(|G|)$, the logarithm of the size of the network. We will see that our model also shares this property as well as the power law decay and the hierarchical structure, combining all these important features.
In this section we calculate the average length of shortest path between two nodes in G_n. First we give a deterministic way to construct one of the shortest paths between any two nodes in the graph. To do so, we need to introduce some notation. Recall that the graph G is a bipartite graph with partition V_1, V_2, see the beginning of Section 2. We remind

the reader that for $\underline{x}, \underline{y} \in \Sigma_n$, typ$(\underline{x})$, the common prefix $\underline{x} \wedge \underline{y}$ and the postfixes $\tilde{\underline{x}}, \tilde{\underline{y}}$ were defined in Definition 2.1.

Definition 3.4.
For two arbitrary vertices $\underline{x}, \underline{y} \in \Sigma_n$ we denote the length of their common prefix by $k = k(\underline{x}, \underline{y}) := |\underline{x} \wedge \underline{y}|$. Furthermore, let us decompose the postfixes $\tilde{\underline{x}}, \tilde{\underline{y}}$ into blocks of digits of the same type:

$$\tilde{\underline{x}} = \underline{b}_1\underline{b}_2 \dots \underline{b}_r, \ \tilde{\underline{y}} = \underline{c}_1\underline{c}_2 \dots \underline{c}_q, \tag{18}$$

such that all of the blocks have a nonzero type and the consecutive blocks are of different types. That is, for $i = 1, \dots, r-1, \ j = 1, \dots q-1$ we have

$$typ(b_i) \neq typ(b_{i+1}) \in \{1, 2\}, \ \text{and } typ(c_j) \neq typ(c_{j+1}) \in \{1, 2\}.$$

Note, that we denoted the number of blocks in $\tilde{\underline{x}}, \tilde{\underline{y}}$ by r and q, respectively. If $\underline{X}$ and $\underline{Y}$ are two random vertices of G_n, then the same notation as in (18) is used with capital letters.

Now we fix an arbitrary self-map p of Σ_1 such that

$$(x, p(x)) \in E(G) \ \forall x \in G.$$

Most commonly, $p(p(x)) \neq x$. Note that x and $p(x)$ have different types since G is bipartite. For a word $\underline{z} = (z_1 \dots z_m)$ with typ$(\underline{z}) \in \{1, 2\}$ we define $p(\underline{z}) := (p(z_1) \dots p(z_m))$.. Then,

$$(\underline{tz}, \underline{tp}(\underline{z})) \text{ is an edge in } G_{\ell+m}, \forall \underline{t} = (t_1 \dots t_\ell), \tag{19}$$

follows from (1).
As usual we write Diam(G) for the maximal graph-distance in the graph G within components of G. Clearly Diam$(G) \leq N - 1$.

Lemma 3.5. *Let $\underline{x}, \underline{y}$ be arbitrary vertices in the same connected component of G_n. Using the notation above, the length of the shortest path between them is at least $r + q - 1$ and at most $r + q + Diam(G) - 2$.*

Considering the worst case scenario, i.e. choosing all blocks of length 1 yields:

Corollary 3.6. *The diameter of the graph G_n is at most $2n + Diam(G) - 2$. Since the size of the graph is N^n, therefore*

$$Diam(G_n) = \frac{2}{\log N} \log(|G_n|) + O(1).$$

Proof of Lemma 3.5. First we construct a path $P(\underline{x},\underline{y})$ of minimal length. Starting from $\underline{x}$ the first half of the path $P(\underline{x},\underline{y})$ is as follows:

$$\begin{aligned}\hat{\underline{x}}^0 &= \underline{x} = (\underline{x}\wedge\underline{y})\underline{b}_1\dots\underline{b}_{r-1}\underline{b}_r\\ \hat{\underline{x}}^1 &= (\underline{x}\wedge\underline{y})\underline{b}_1\dots\underline{b}_{r-1}p(\underline{b}_r)\\ &\dots\\ \hat{\underline{x}}^{r-1} &= (\underline{x}\wedge\underline{y})\underline{b}_1p(\underline{b}_2\dots p(\underline{b}_{r-1}p(\underline{b}_r))),\end{aligned}$$

Starting from $\underline{y}$ the first half of the path $P(\underline{x},\underline{y})$ is as follows:

$$\begin{aligned}\hat{\underline{y}}^0 &= \underline{y} = (\underline{x}\wedge\underline{y})\underline{c}_1\underline{c}_2\dots\underline{c}_r\\ \hat{\underline{y}}^1 &= (\underline{x}\wedge\underline{y})\underline{c}_1\dots\underline{c}_{r-1}p(\underline{c}_r)\\ &\dots\\ \hat{\underline{y}}^{q-1} &= (\underline{x}\wedge\underline{y})\underline{c}_1p(\underline{c}_2\dots p(\underline{c}_{r-1}p(\underline{c}_q))).\end{aligned}$$

It follows from (19) that

$$\begin{aligned}P_x &:= (\hat{\underline{x}}^0,\hat{\underline{x}}^1,\dots,\hat{\underline{x}}^{r-1})\\ P_y &:= (\hat{\underline{y}}^{q-1},\cdots\hat{\underline{y}}^1,\hat{\underline{y}}^0)\end{aligned}$$

are two paths in G_n. To construct $P(\underline{x},\underline{y})$ the only thing remained is to connect $\hat{\underline{x}}^{r-1}$ and $\hat{\underline{y}}^{q-1}$. Using (19) it is easy to see that this can be done with a path P_c of length at most $\mathrm{Diam}(G)$. In this way,

$$P(\underline{x},\underline{y}) := P_xP_cP_y.$$

Clearly,

$$r+q-1\le \mathrm{Length}(P(\underline{x},\underline{y}))\le r+q+\mathrm{Diam}(G)-2$$

On the other hand, now we prove that no shorter paths exists than $P(\underline{x},\underline{y})$. Recall that it follows from (1) that for any path $Q(\underline{x},\underline{y}) = (\underline{x}=\underline{q}^0,\dots,\underline{q}^\ell=\underline{y})$, the consecutive elements of the path only differ in their postfixes, which have different types. That is,

$$\forall i,\underline{q}^i=\underline{w}^i\underline{z}^i,\ \underline{q}^{i+1}=\underline{w}^i\tilde{\underline{z}}^i,\ \text{with } \mathrm{typ}(\underline{z}^i)\neq\mathrm{typ}(\tilde{\underline{z}}^i)\in\{1,2\}.$$

This implies that in each step on the path, the number of blocks in (18) changes by at most one. Recall that $|\underline{x}\wedge\underline{y}|=k$, so $x_{k+1}\neq y_{k+1}$. Since the digit on the $k+1$-th position changes on the path, we have to reach a point where all the digits to the right from the k-th position are of the same type. Starting from $\tilde{\underline{p}}^0=\underline{x}$, to reach the first vertex $\underline{a}$ of this property, we need at least $r-1$ steps on any path $\tilde{P}$, where r was defined in formula (18). Similarly, starting from $\underline{y}$, we need at least $q-1$ steps to reach the first vertex $\underline{b}$ where all the digits after the

k-th position are of the same type. Because $x_{k+1} \neq y_{k+1}$, we need at least one more edge and at most $\mathrm{Diam}(G)$ edges. □

Theorem 3.7. *The expectation of the length of a shortest path between two uniformly chosen vertices $\underline{X}, \underline{Y} \in G_n$ can be bounded by*

$$\frac{4n_1n_2}{N^2}(n-1) < \mathbb{E}(|P(\underline{X}, \underline{Y})|) < N + \frac{4n_1n_2}{N^2}(n-1).$$

Corollary 3.8. *The magnitude of the average length of a shortest path between two uniformly chosen vertices in G_n is the logarithm of the size of G_n, which is the same order as Diam(G_n).*

Proof of Theorem 3.7. Let $\underline{X}, \underline{Y}$ be independent, uniformly chosen vertices of G_n. In this proof we use the notation introduced in Definitions 2.1 and 3.4. The digits of the code of a uniformly chosen vertex are independent and uniform in $\{0, \ldots, N-1\}$, hence $K(\underline{X}, \underline{Y}) := |\underline{X} \wedge \underline{Y}|$ has a truncated geometric distribution with parameter $\frac{N-1}{N}$. That is

$$\mathbb{P}(K(\underline{X}, \underline{Y}) = k) = \begin{cases} \left(\frac{1}{N}\right)^k \cdot \frac{N-1}{N}, & \text{if } 0 \leq k < n, \\ \left(\frac{1}{N}\right)^n & \text{if } k = n. \end{cases}$$

Furthermore, given that the length of the prefix is $k = K(\underline{X}, \underline{Y})$, the random variables R and Q (see Definition 2.1) can be represented as the sum of indicators corresponding to the start of a new block:

$$R = 1 + \sum_{i=1}^{n-k-1} \mathbb{1}_{\mathrm{typ}(X_{k+i}) \neq \mathrm{typ}(X_{k+i+1})},$$
$$Q = 1 + \sum_{i=1}^{n-k-1} \mathbb{1}_{\mathrm{typ}(Y_{k+i}) \neq \mathrm{typ}(Y_{k+i+1})}.$$

Taking expectation yields

$$\begin{aligned} \mathbb{E}(Q|K(\underline{X}, \underline{Y}) = k) &= \mathbb{E}(R|K(\underline{X}, \underline{Y}) = k) \\ &= 1 + \mathbb{E}\left(\sum_{i=1}^{n-k-1} \mathbb{1}_{\mathrm{typ}(X_{k+i}) \neq \mathrm{typ}(X_{k+i+1})}\right) \\ &= 1 + \sum_{i=1}^{n-k-1} \mathbb{P}(\mathrm{typ}(X_{k+i}) \neq \mathrm{typ}(X_{k+i+1})) \\ &= 1 + (n-k-1)\frac{2n_1n_2}{N^2}. \end{aligned}$$

So weighting this with the geometric weights of the length of the prefix, we get

$$\begin{aligned}\mathbb{E}(Q) = \mathbb{E}(R) &= \mathbb{E}\big(\mathbb{E}(R|K(\underline{X},\underline{Y}))\big)\\ &= \mathbb{E}\big(1+(n-K(\underline{X},\underline{Y})-1)\frac{2n_1n_2}{N^2}\big)\\ &= 1+\left(n-\frac{1}{N-1}\left(1-\frac{1}{N^n}\right)-1\right)\frac{2n_1n_2}{N^2}.\end{aligned}$$

Using this and the following immediate consequence of Lemma 3.5

$$-1 \le \mathbb{E}(|P(\underline{X},\underline{Y})| - (R+Q)) \le \mathrm{Diam}(G) - 2,$$

finally we obtain that

$$1-\frac{1}{N-1}+\frac{4n_1n_2}{N^2}(n-1) \le \mathbb{E}(|P(\underline{X},\underline{Y})| < \mathrm{Diam}(G)+\frac{4n_1n_2}{N^2}(n-1).$$

□

3.4. **Decay of local clustering coefficient of the modified sequence $\left\{\hat{G}_n\right\}$.** An important property of most real networks is the high degree of clustering. In general, the local clustering coefficient of a node v having n_v neighbors is defined as

$$C_v := \frac{\#\{\text{links between neighbors of } v\}}{\binom{n_v}{2}}.$$

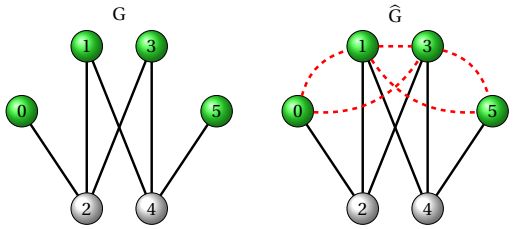
Note that the numerator in the formula is the number of triangles containing v and C_v is the portion of the pairs of neighbors of v which form a triangle with v in the graph.
Observe that without the loops the graph sequence G_n is bipartite, i.e. there are no triangles in the graph G_n. However, we can modify the graph sequence G_n in a natural way, like in [2], to get a new sequence $\hat{G}_n$ preserving the hierarchical structure of G_n, still reflecting the dependence of clustering coefficient on node degree observed in several real networks. Namely, the local clustering coefficient of a vertex v is of order $1/\deg(v)$.

Definition 3.9.

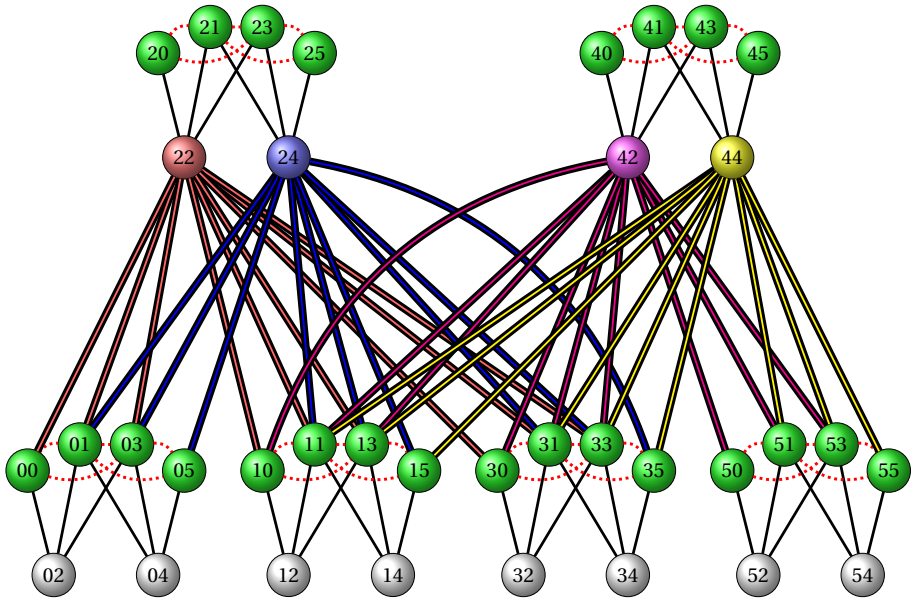
- *We obtain the graph $\hat{G}$ adding a set of extra edges $RE(\hat{G})$ to G satisfying the following property:*

  Property R

  $\forall x \in \Sigma_1$, $\exists y, z \in \Sigma_1$, such that two among the edges of the triangle $(x,y,z)_\Delta$ are contained in $E(G)$ and one of the edges is in $RE(\hat{G})$.



(a) We obtain $\widehat{G}$ by adding the dashed (red) edges to G.



(b) $\widehat{G}_2$: The edges of $\widehat{G}_2$ and G_2 differ only at the lowest hierarchical level (cf. Figure 3)

FIGURE 4. Clustering extended "fan".

So,

$$V(\hat{G}) = V(G) \text{ and } E(\hat{G}) = E(G) \cup RE(\hat{G}).$$

In the example presented on Figure 4 the edges from $RE(\hat{G})$ are the dashed red edges.

- *Similarly we define the graph sequence $\left\{\hat{G}_n\right\}_{n=1}^{\infty}$ by deleting all loops in G_n and adding extra edges to G_n. That is, the vertices $V(\hat{G}_n) = V(G_n) = \Sigma_n$, and with the definition of the simple graph G'_n in Section 2.5, the edge set is extended by the following*

rule

$$E(\hat{G}_n) = E(G'_n) \bigcup RE(\hat{G}_n), \tag{20}$$

where

$$RE(\hat{G}_n) = \left\{ \begin{pmatrix} x_1 \dots x_n \\ y_1 \dots y_n \end{pmatrix} : x_i = y_i, i \leq n-1, \begin{pmatrix} x_n \\ y_n \end{pmatrix} \in RE(\hat{G}) \right\}. \tag{21}$$

It is clear from Property **R** that

$$\hat{C}_{\min} := \min_{x \in \hat{G}} C_x > 0. \tag{22}$$

Further, using (1) and (21) one can easily see that the degree of a vertex $\underline{x} \in \hat{G}_n$ is

$$\widehat{\deg}_n(\underline{x}) = S(\underline{x}) \cdot \deg(x_n) + \left(\widehat{\deg}(x_n) - \deg(x_n) \right), \tag{23}$$

where $\widehat{\deg}(.)$ denotes the degree of a vertex in $\hat{G}$, while $\deg(.)$ stands for the degree in G.

Remark 3.10. *The difference between the degree of any node $\underline{x} \in \Sigma_n$ in G_n and in $\hat{G}_n$ is bounded, thus the degree sequence of $\hat{G}_n$ has the same power law exponent as G_n.*

Theorem 3.11. *There exists $K_1, K_2 > 0$ such that the local clustering coefficient $C_{\underline{x}}$ of an arbitrary node $\underline{x} \in \hat{G}_n$ satisfies.*

$$\frac{K_1}{\widehat{\deg}_n(\underline{x})} \leq C_{\underline{x}} \leq \frac{K_2}{\widehat{\deg}_n(\underline{x})}.$$

Proof. We write $\mathcal{T}_n(\underline{x})$ for the set of all triangles in $\hat{G}_n$ containing the node $\underline{x} \in \Sigma_n$. We say that a triangle $(\underline{x}, \underline{y}, \underline{z})_\Delta \in \mathcal{T}_n(\underline{x})$ is regular if and only if exactly two of its edges are from $E(G_n)$. The triangle $(\underline{x}, \underline{y}, \underline{z})_\Delta \in \mathcal{T}_n(\underline{x})$ is called irregular if it is not regular. The set of irregular triangles containing $\underline{x}$ is denoted by $\mathcal{IRT}_n(\underline{x})$. We partition the set of regular triangles $\mathcal{RT}_n(\underline{x})$ into the classes:

$$\mathcal{RT}_n(\underline{x}) = \mathcal{RT}^1_n(\underline{x}) \cup \mathcal{RT}^2_n(\underline{x})$$

in the following way: A triangle $(\underline{x}, \underline{y}, \underline{z})_\Delta \in \mathcal{RT}_n(\underline{x})$ belongs to $\mathcal{RT}^1_n(\underline{x})$ if and only if $\underline{x}$ is NOT an endpoint of the edge contained in $RE(\hat{G}_n)$. That is

$$\mathcal{RT}^1_n(\underline{x}) := \left\{ (\underline{x}, \underline{y}, \underline{z})_\Delta \in \mathcal{RT}_n(\underline{x}) : \begin{pmatrix} \underline{x} \\ \underline{y} \end{pmatrix}, \begin{pmatrix} \underline{x} \\ \underline{z} \end{pmatrix} \in E(G_n). \right\}$$

Hence, $\mathcal{RT}^2_n(\underline{x})$ is the set of those $(\underline{x},\underline{y},\underline{z})_\Delta \in \mathcal{RT}_n(\underline{x})$ for which either $\binom{\underline{x}}{\underline{y}} \in E(G_n)$ and $\binom{\underline{x}}{\underline{z}} \in RE(\hat{G}_n)$ or vice versa. Summarizing these partitions:

$$\mathcal{T}_n(\underline{x}) = \mathcal{RT}_n(\underline{x}) \cup \mathcal{IRT}_n(\underline{x}) = \mathcal{RT}^1_n(\underline{x}) \cup \mathcal{RT}^2_n(\underline{x}) \cup \mathcal{IRT}_n(\underline{x})$$

Now we define the cardinality of these classes:

$$\Delta^1_n(\underline{x}) := \#\mathcal{RT}^1(\underline{x}), \Delta^2_n(\underline{x}) := \#\mathcal{RT}^2(\underline{x}) \text{ and } \Delta^{\mathrm{ir}}_n(\underline{x}) := \#\mathcal{IRT}(\underline{x}).$$

When $n = 1$ then we suppress the index n. Observe that by Property **R**,

$$\Delta^{\mathrm{r}}_n(\underline{x}) := \Delta^1_n(\underline{x}) + \Delta^2_n(\underline{x}) \geq 1, \quad \forall n \geq 1, \underline{x} \in \Sigma_n.$$

Now we compute $\Delta^i_n(\underline{x})$, $i \in \{1, 2, \mathrm{ir}\}$, for an arbitrary fixed $\underline{x} \in \Sigma_n$. To do so the notation $\ell(\underline{x})$ will be used. First we verify that

$$\Delta^1_n(\underline{x}) = \sum_{r=0}^{\ell(\underline{x})-1} \prod_{j=1}^{r} \deg(x_{n-j}) \cdot \Delta^1(x_n) = S(\underline{x}) \cdot \Delta^1(x_n), \tag{24}$$

where $S(\underline{x})$ was defined in (3). To see this, observe that it follows from (1), (20) and (21) that

$$(\underline{x},\underline{y},\underline{z})_\Delta \in \mathcal{RT}^1_n(\underline{x})$$

holds if and only if all of the following three assertions are satisfied:

(1) $\exists 0 \leq r \leq \ell(\underline{x})-1$, $|\underline{y} \wedge \underline{z}| = n-1$ and $|\underline{x} \wedge \underline{y}| = |\underline{x} \wedge \underline{z}| = n-r-1$
(2) $\binom{x_k}{y_k} \in E(G)$ whenever $n - r \leq k \leq n-1$
(3) $(x_n, y_n, z_n)_\Delta \in \mathcal{RT}^1(x_n)$.

Hence (24) is obtained by an immediate calculation.
Now we prove that

$$\Delta^2_n(\underline{x}) = \sum_{r=0}^{\ell(\underline{x})-1} \prod_{j=1}^{r} \deg(x_{n-j}) \cdot \Delta^2(x_n) = S(\underline{x}) \cdot \Delta^2(x_n). \tag{25}$$

This is so because by (1), (20) and (21) we have

$$(\underline{x},\underline{y},\underline{z})_\Delta \in \mathcal{RT}^2_n(\underline{x})$$

holds if and only if all of the following three assertions are satisfied:

(1) $\exists 0 \leq r \leq \ell(\underline{x})-1$, $|\underline{x} \wedge \underline{y}| = n-1$ and $|\underline{x} \wedge \underline{z}| = |\underline{y} \wedge \underline{z}| = n-r-1$,
(2) $\binom{x_k}{z_k} \in E(G)$ whenever $n - r \leq k \leq n-1$
(3) $(x_n, y_n, z_n)_\Delta \in \mathcal{RT}^2(x_n)$.

Hence, using the same argument as above we get (25).
Finally, we determine the number of irregular triangles containing $\underline{x}$:

$$\Delta^{\mathrm{ir}}_n(\underline{x}) = \Delta^{\mathrm{ir}}(x_n). \tag{26}$$

This follows from the fact that

$$(\underline{x},\underline{y},\underline{z})_\Delta \in \mathcal{IRT}_n(\underline{x})$$

is equivalent to

$$\forall 1 \le i \le n-1,\ x_i = y_i = z_i \text{ and } (x_n, y_n, z_n)_\Delta \in \mathcal{IRT}(x_n).$$

We write $Z_\Delta(\underline{x})$ for the number of all triangles in $\hat{G}_n$ containing $\underline{x}$:

$$Z_\Delta(\underline{x}) := \underbrace{\Delta_n^1(\underline{x}) + \Delta_n^2(\underline{x})}_{\Delta^{\mathrm{r}}(\underline{x})} + \Delta_n^{\mathrm{ir}}(\underline{x}).$$

Using (23), (24), (25) and (26) we get

$$C_{\underline{x}} = \frac{Z_\Delta(\underline{x})}{\binom{\widehat{\deg}_n(\underline{x})}{2}} = \frac{2\Delta^{\mathrm{r}}(x_n)\cdot S(\underline{x}) + 2\Delta^{\mathrm{ir}}(x_n)}{\widehat{\deg}_n(\underline{x})(\widehat{\deg}_n(\underline{x})-1)}, \tag{27}$$

where $S(\underline{x})$ was defined in (3). Now we estimate $C_{\underline{x}}$.

Claim 3.12.

(i): *If $\ell(\underline{x}) = 1$, then $C_{\underline{x}} = C_{x_n}$.*
(ii): *If $\ell(\underline{x}) \ge 2$, then we have*

$$\left| C_{\underline{x}} - \frac{2\Delta^{\mathrm{r}}(x_n)}{\deg(x_n)} \cdot \frac{1}{\widehat{\deg}_n(\underline{x})} \right| \le \frac{\mathrm{const}}{\widehat{\deg}_n^{\,2}(\underline{x})}. \tag{28}$$

Proof of the Claim. Part **(i)** immediately follows from (1). To prove **(ii)** we fix an arbitrary $\underline{x} \in \Sigma_n$ with $\ell(\underline{x}) \ge 2$. Since t, u, v introduced below depend only on x_n there exists a constant C_* independent of n and $\underline{x}$ such that

(29)

$$0 \le t := \frac{\Delta^r(\underline{x})}{\deg(x_n)}, \quad u := \widehat{\deg}(x_n) - \deg(x_n), \quad v := 2\Delta^{\mathrm{ir}}(x_n) < C_*.$$

To prove (28) it is enough to verify that

$$Q := \left(\widehat{\deg}_n(\underline{x})\right)\left(\widehat{\deg}_n(\underline{x})-1\right)\cdot C_{\underline{x}} - 2t\cdot(\widehat{\deg}_n(\underline{x})-1)$$

is bounded in n and $\underline{x} \in \Sigma_n$. This so, because by (23) and (27) we have

$$\begin{aligned} Q &= 2\Delta^r(x_n)\cdot S + v - 2t\big(\underbrace{S\cdot\deg(x_n)+u}_{\widehat{\deg}(\underline{x})} - 1\big) \\ &= 2\Delta^r(x_n)\cdot S + v - \underbrace{2\Delta^r(x_n)\cdot S}_{2tS\deg(x_n)} - 2t(u-1) \\ &= v - 2t(u-1), \end{aligned}$$

which is bounded by (29). □

Property **R** implies that both C_{x_n} and $\frac{\Delta^{\mathrm{r}}(x_n)}{\deg(x_n)}$ are bounded away from zero. This completes the proof of the Theorem 3.11. □

The following theorem shows that the graph sequence $\hat{G}_n$ displays similar features to that of considered in [2], namely, the average local clustering coefficient of the graphs $\hat{G}_n$ is not tending to zero with the size of $\hat{G}_n$.

Theorem 3.13. *The average local clustering coefficient $\bar{C}(\hat{G}_n)$ of the graph $\hat{G}_n$ is bounded by two positive constants, more precisely*

$$\frac{2n_1n_2\hat{C}_{\min}}{N^2} \le \bar{C}(\hat{G}_n) \le \bar{C}(\hat{G}), \tag{30}$$

where $\hat{C}_{\min}$ was defined in (22).

Proof. We will use the notation introduced in the proof of Theorem 3.11. It easily follows from the proof of Theorem 3.11 that

$$C_{\underline{x}} \le C_{x_n}. \tag{31}$$

Namely, if $\ell(\underline{x}) = 1$ then by (21), $C_{\underline{x}} = C_{x_n}$. If $\ell(\underline{x}) \ge 2$ then $S(\underline{x}) \ge 1$ thus using (27) we obtain

$$\begin{aligned} C_{\underline{x}} &\le \underbrace{\frac{\Delta^{\mathrm{r}}(x_n) + \Delta^{\mathrm{ir}}(x_n)}{\binom{\deg(x_n)}{2}}}_{C_{x_n}} \cdot \underbrace{\binom{\deg(x_n)}{2} \cdot \frac{S(\underline{x})}{\binom{\widehat{\deg_n}(\underline{x})}{2}}}_{\le 1} \cdot \\ &\le C_{x_n}. \end{aligned}$$

This completes the proof of (31) from which the upper estimate of (30) follows by averaging. On the other hand to see that the lower estimate holds we take into consideration only the contribution of $\underline{x} \in \Sigma_n$ with $\ell(\underline{x}) = 1$.

$$\bar{C}(\hat{G}_n) > \frac{1}{N^n}\left(\sum_{z\in V_1} N^{n-2}n_2C_z + \sum_{z\in V_2} N^{n-2}n_1C_z\right)$$

Using $C_z > \hat{C}_{\min}$, the lower bound of (30) follows. □

4. Definition of the randomized model

In this section we randomize the deterministic model in Section 2 by using Λ in $[0,1]^2$. The random graph sequence G_n^{r} is generated in a way which was inspired by the W-random graphs introduced by Lovász and Szegedy [10]. See also [5].

Fix a deterministic model with a base graph G, $|V(G)| = N$. This determines $\Lambda(a,b)$ the limit of the sequence of scaled adjacency matrices,

see the definition (7) and (6) in Section 2.2. Now for each n, we throw $M_n + 1$ independent, uniform random numbers over $[0, 1]$:

$$X^{(1)}, X^{(2)}, \dots, X^{(M_n+1)} \sim U[0,1], \text{ i.i.d.}$$

We denote the N-adic expansion of each of these numbers by

$$\underline{X}^{(i)} = (X_1^i, X_2^i, \dots), \quad \text{i.e. } X^{(i)} = \sum_{k=1}^{\infty} \frac{X_k^i}{N^k},$$

where the X_k^i-s are uniform over the set $\{0, 1, \dots, N-1\}$. The n-th approximation of $X^{(i)}$ is

$$X_{[n]}^{(i)} = \sum_{k=1}^{n} \frac{X_k^i}{N^k}, \quad \underline{X}_n^{(i)} = (X_1^i, \dots, X_n^i).$$

Now we construct the random graph G_n^{r} as follows: $|V(G_n^{\mathrm{r}})| = \{1, \dots, M_n\}$, and $E(G_n^{\mathrm{r}})$ is given by

$$E(G_n^{\mathrm{r}}) = \left\{ (i,j) \middle| \text{ int} \left(I_{X_{[n]}^{(i)}} \times I_{X_{[n]}^{(j)}} \right) \cap \Lambda \neq \emptyset \right\},$$

where int denotes the interior of a set. Clearly,

$$E(G_n^{\mathrm{r}}) = \left\{ (i,j) | \Lambda_n(X^{(i)}, X^{(j)}) = 1 \right\}.$$

Note that

$$\Lambda_n(X^{(i)}, X^{(j)}) = 1 \;\Leftrightarrow\; \begin{pmatrix} X_1^i \dots X_n^i \\ X_1^j \dots X_n^j \end{pmatrix} \in E(G_n).$$

Namely, we can think of the first n digits $(X_1^i, \dots, X_n^i)$ and $(X_1^j, \dots, X_n^j)$ of the N-adic expansion of $X^{(i)}$ and $X^{(j)}$ as vertices in G_n. We draw an edge between the two vertices i and j in G_n^{r} if the vertices $(X_1^i \dots X_n^i)$ and $(X_1^j \dots X_n^j)$ are connected by an edge in the deterministic model G_n. This gives the following probabilistic interpretation of the random model:

Remark 4.1. *Consider the deterministic graph sequence G_n with urns sitting at each vertex $v \in G_n$. Now throw $M_n + 1$ balls independently and uniformly into the urns, and connect vertex i to vertex j by an edge in the random graph G_n^{r} if and only if the urns of ball i and j are connected by and edge in G_n.*

We need to introduce some further notation.

Frequently used definitions. Under assumption **A1**, for an $\underline{x} \in G_n$ with $\ell(\underline{x}) = k$ the degree of $\underline{x}$ is

$$t_k := \frac{d_1^{k+1} - 1}{d_1 - 1} + 1,$$

independently of the length of n.
In the random graph G_n^{r}, the conditional probability of the degree distribution of a random node $V \in \{0, \dots, M_n\}$ conditioned on the first n digits of the N-adic expansion of the corresponding code $X^{(V)}$ follows a Binomial distribution:

$$\left(\deg(V)|(X_1^V \dots X_n^V) = \underline{x}\right) \sim BIN\left(M_n, \frac{t_{\ell(\underline{x})}}{N^n}\right). \tag{32}$$

This follows from the characterization of G_n^{r} described in Remark 4.1. Namely, assume that the V-th ball has landed in urn with label $\underline{x} \in \Sigma_n$. In G_n there are exactly $\deg_n(\underline{x}) - 1 = t_{\ell(\underline{x})}$ vertices $\underline{y} \in \Sigma_n$ that are connected to $\underline{x}$. All the balls landing into urns corresponding to these vertices $\underline{y}$ will be connected to V in G_n^{r}.

5. Properties of the randomized model

In this section we determine the proportion of isolated vertices and characterize the degree sequence.

5.1. **Isolated vertices.**

Theorem 5.1. *If $M_n = c_n N^n$ with $\lim_{n\to\infty} c_n = \infty$, then the fraction of isolated vertices tends to zero as $n \to \infty$. More precisely, for a uniformly chosen node $V \in G_n^r$,*

$$\mathbb{P}\left(\deg(V) = 0\right) \le e^{-d_{\min} c_n},$$

where $d_{\min}$ stands for the minimal degree in the base graph G, and in deg(.) *we do not count the loops.*

The following corollary is an immediate consequence of the Borel-Cantelli lemma.

Corollary 5.2. *If $\sum_{n=1}^{\infty} c_n N^n e^{-d_{\min} c_n} < \infty$, then almost surely there will be only finitely many n-s, for which the graph G_n^{r} has isolated vertices.*

The assumption of the Corollary is satisfied if e.g. $c_n > n \log(N+1)$.

Proof of Theorem 5.1. Given the N-adic expansion of $X^{(V)}$, the probability that a vertex is isolated depends on how many neighbors the vertex $(X_1^V \dots X_n^V)$ has in the deterministic model. So we can write

$$\mathbb{P}\left(\deg(V) = 0\right) = \sum_{\underline{x} \in \Sigma_n} \mathbb{P}(\deg(V) = 0 | (X_1^V \dots X_n^V) = \underline{x}) \cdot \frac{1}{N^n}$$

As we have already seen, $\big(\deg(V)|(X_1^V \dots X_n^V) = \underline{x}\big)$ follows a Binomial distribution with parameters M_n and $\frac{\deg_n(\underline{x})-1}{N^n}$, so the conditional probability of isolation is

$$\begin{aligned}\mathbb{P}(\deg(V)=0|(X_1^V \dots X_n^V)=\underline{x}) &= \left(1-\frac{t_{\ell(\underline{x})}}{N^n}\right)^{M_n} \\ &\le e^{-\deg_n(\underline{x})c_n}(1+o(1)).\end{aligned}$$

Obviously $e^{-\deg_n(\underline{x})c_n} \le e^{-d_{\min}c_n}$ holds for all $\underline{x} \in \Sigma_n$, which completes the proof. □

5.2. **Decay of degree distribution.** Fix a constant K such that for a standard normal variable Z, $\mathbb{P}(|Z| > K) < e^{-10}$. We write

$$I_{k,n} := [c_n t_k - K\sqrt{c_n t_k}, c_n t_k + K\sqrt{c_n t_k}],$$

and

$$k_0(n) := \max\left\{(n+1)\frac{\log d_2}{\log d_1}, \frac{\log n}{\log d_1}\right\}.$$

Now we describe the degree distribution for the random model.

Theorem 5.3. *Let $k > k_0(n)$ and $u \in I_{k,n}$. Then for a uniformly chosen node V in G_n^r*

$$\mathbb{P}\left(\deg(V)=u\right) = \Big(\frac{n_1}{N}\Big)^k \frac{n_2}{N}\cdot\frac{1}{\sqrt{c_n t_k}}\phi\Big(\frac{u-c_n t_k}{\sqrt{c_n t_k(1-\frac{t_k}{N^n})}}\Big)\big(1+O(\frac{1}{\sqrt{c_n t_k}})\big),$$

where ϕ denotes the density function of a standard Gaussian variable.

This immediately implies

Corollary 5.4. *The degree distribution of the random model is given by the following formula for $a, b \in [-K, K]$:*

$$\begin{aligned}\mathbb{P}\left(\deg(V) \in [c_n t_k + a\sqrt{c_n t_k}, c_n t_k + b\sqrt{c_n t_k}]\right) &= \Big(\frac{n_1}{N}\Big)^k \frac{n_2}{N}\cdot(\Phi(b)-\Phi(a)) \\ &\quad + O\Big(\Big(\frac{n_1}{N}\Big)^k \frac{1}{\sqrt{c_n t_k}}\Big),\end{aligned}$$

where $k > k_0(n)$ and Φ denotes the distribution function of a standard Gaussian variable. So, for $u \in I_{k,n}$, $k > k_0(n)$ the tail of the probability

distribution is:
(33)

$$\mathbb{P}(\deg(V) > u) = \Big(\frac{n_1}{N}\Big)^{k+1} + \Big(\frac{n_1}{N}\Big)^{k} \frac{n_2}{N} \left(1 - \Phi\Big(\frac{u - c_n t_k}{\sqrt{c_n t_k (1 - \frac{t_k}{N^n})}}\Big)\right) + \Big(\frac{n_1}{N}\Big)^{k+1} O\Big(\frac{1}{\sqrt{c_n t_k}}\Big).$$

This holds because $\mathbb{P}(\deg(V) > u)$ equals the sum of all probability mass that is concentrated around t_l-s for $l \geq k+1$, resulting in the first term, plus the second term coming from the part greater than u of the binomial mass around t_k. As a consequence, the decay of the degree distribution follows a power law. Namely, the following holds

Theorem 5.5. *Let*

$$\gamma := 1 + \frac{\log(\frac{N}{n_1})}{\log d_1}.$$

Then the decay of the degree distribution is:

$$\mathbb{P}(\deg(V) > u) = u^{-\gamma+1} \cdot L(u),$$

where $L(u)$ is a bounded function:

$$\frac{n_1}{N} \leq L(u) \leq \frac{N}{n_1}.$$

The idea of the proof of Theorem 5.3. The conditional distribution of the degree of a node V conditioned on the n-digit N-adic expansion of $X_n^{(V)} = \underline{x}$ follows a $BIN(c_n N^n, \frac{t_{\ell(\underline{x})}}{N^n})$ law. This is close to a $POI(c_n t_{\ell(\underline{x})})$ random variable, because c_n and $t_{\ell(\underline{x})}$ tend to infinity in a much smaller order than N^n. Now for the $POI(c_n t_{\ell(\underline{x})})$ variable, the Central Limit Theorem holds with an error term of order $1/\sqrt{c_n t_{\ell(\underline{x})}}$. Now the unconditional degree distribution comes from the law of total probability and from the fact that all other errors are negligible. □

Proof of Theorem 5.3. We determined the degree distribution of the deterministic model under assumption (**A1**), see Section 3.1 for details. Recall that if $k > k_0(n)$, then the mass at t_k is

$$p_k := \mathbb{P}(\ell(\underline{x}) = k) = \Big(\frac{n_1}{N}\Big)^{k} \frac{n_2}{N}.$$

We show that in the random model G_n^{r}, these Dirac masses are turned into Gaussian masses centered at $c_n t_k$. Suppose $u \in I_{k,n}$. By the law

of total probability, we have

(34)
$$\mathbb{P}(\deg(V)=u) = \mathbb{P}(\deg(V)=u|\left(X_1^V \dots X_n^V\right)=\underline{x}, \ell(\underline{x})=k)\cdot p_k + S_1 + S_2,$$

where

$$S_1 = \sum_{j=1}^{k-1} \mathbb{P}(\deg(V)=u|\left(X_1^V \dots X_n^V\right)=\underline{x}, \ell(\underline{x})=j)\cdot p_j$$

$$S_2 = \sum_{j=k+1}^{n} \mathbb{P}(\deg(V)=u|\left(X_1^V \dots X_n^V\right)=\underline{x}, \ell(\underline{x})=j)\cdot p_j$$

S_1 and S_2 combines the total contribution of cases when $\ell(X_1^V \dots X_n^V) \neq k$, i.e. referring to the urn model of our random graph, S_1+S_2 settles the cases when the random ball V falls into an urn which has degree different from t_k in G_n. As a first step in our proof we show that the right hand side in the first line of (34) gives the formula in Theorem 5.3, then as a second step we verify that S_1+S_2 is negligible.

First step: Following the standard proof of the local form of de Moivre-Laplace CLT, we obtain that for $u \in I_{k,n}$

$$\mathbb{P}\Big(\deg(V)=u|\left(X_1^V \dots X_n^V\right)=\underline{x}\Big) = \frac{1}{\sqrt{c_n t_{\ell(\underline{x})}(1-\frac{t_{\ell(\underline{x})}}{N^n})}}\phi\left(\frac{u-c_n t_{\ell(\underline{x})}}{\sqrt{c_n t_{\ell(\underline{x})}(1-\frac{t_{\ell(\underline{x})}}{N^n})}}\right)\cdot\left(1+O(\frac{1}{\sqrt{c_n t_{\ell(\underline{x})}}})\right).$$

We can neglect $1-\frac{t_{\ell(\underline{x})}}{N^n}$. This completes the first step.

Second step: Since $u \in I_{k,n}$ we have:

(35)
$$S_1 \leq \sum_{j=1}^{k-1} \mathbb{P}(\deg(V) > t_k - K\sqrt{t_k}|(X_1^V \dots X_n^V)=\underline{x}, \ell(\underline{x})=j)\cdot p_j$$

$$S_2 \leq \sum_{j=k+1}^{n} \mathbb{P}(\deg(V) < t_k + K\sqrt{t_k}|(X_1^V \dots X_n^V)=\underline{x}, \ell(\underline{x})=j)\cdot p_j$$

Now we use the fact known from Chernoff-bounds: for an $Z \sim BIN(m,p)$ variable

$$\mathbb{P}(Z \geq (1+\delta)\mathbb{E}(Z)) \leq e^{-\frac{1}{2}\delta^2\mathbb{E}(Z)},$$

and the same bound holds for $\mathbb{P}(Z \leq (1-\delta)\mathbb{E}(S))$. By (32), to estimate each summand in (35) we can apply these inequalities for

$Z_j \sim BIN(c_n N^n, \frac{t_j}{N^n}),\ j \in \{1,\dots,n\} \setminus \{k\}$, yielding an upper bound

$$\begin{aligned} S_1 + S_2 &\le \sum_{j=1}^{k-1} e^{-\frac{1}{2}d_1^{2k-j}c_n} \cdot p_j + \sum_{j=k+1}^{n} e^{-\frac{1}{2}(1-d_1^{k-j})^2 d_1^j c_n} \cdot p_j \\ &\le e^{-\frac{1}{8}d_1^k c_n}. \end{aligned}$$

Since $e^{-\frac{1}{8}d_1^k c_n} = o(\frac{1}{\sqrt{c_n t_k}})$, the statement of Theorem 5.3 follows. □

Now we are ready to prove the main result of the section.

Proof of Theorem 5.5. If $u \in I_{k,n}$, then

$$u = d_1^k \cdot \left(1 + O\left(\frac{1}{d}\right)\right).$$

Using (33) we obtain that there exists $C(u) \in [\frac{n_1}{N}, 1]$ such that

$$\mathbb{P}(\deg(V) > u) = \left(\frac{n_1}{N}\right)^k C(u).$$

The last two formulas immediately imply the assertion of the Theorem whenever $u \in I_{k,n}$. Actually in this case we have $\frac{n_1}{N} \le L(u) \le 1$. If $u \notin \cup_k I_{n,k}$, then there exists $k = k(u)$ such that $u \in (c_n t_k, c_n t_{k+1})$. By monotonicity of the distribution function we have

$$\mathbb{P}(\deg(V) > c_n t_{k+1}) \le \mathbb{P}(\deg(V) > u) \le \mathbb{P}(\deg(V) > c_n t_k).$$

Applying the theorem for $c_n t_{k+1}$ and $c_n t_k$, we loose a factor of $\frac{N_1}{n}$ in the upper bound of $L(u)$ and the assertion of the Theorem follows. □

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